Math 211 — Spring 2009
Discrete Structures
Final exam, June 12, 2009 — Duration: 2 hours

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	7	8	TOTAL/96

YOUR NAME:

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1	Section 2	Section 3	Section 4
Recitation Th 3:30	Recitation F 8	Recitation Th 11	Recitation Th 5
Ms. Soukiassian	Professor Makdisi	Ms. Soukiassian	Ms. Soukiassian

INSTRUCTIONS:

1. **Write your NAME and AUB ID number, and circle your SECTION above.**
2. **Solve the problems inside the booklet. Explain your reasoning precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points. EACH PART OF A GIVEN PROBLEM CARRIES EQUAL WEIGHT.**
3. **You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.**
4. **Open book and notes. CALCULATORS ARE ALLOWED. Turn OFF and put away any cell phones.**
5. **You do not need to simplify your answers to counting problems — for example, if the answer is $\binom{211}{30} + (12!)7 \cdot 6 \cdot 5$, just leave it that way.**

NEW! 6. The problems are ordered in the order of the topics we covered during the semester, NOT in order of increasing difficulty. Take a few minutes at the beginning of the exam to look through the problems and to decide in what order you want to solve the problems.

NEW! 7. In each problem, the difficulty of the parts varies even though they have equal weight. EVEN IF YOU CANNOT DO AN EARLIER PART OF A PROBLEM, YOU MAY ASSUME THE RESULT FOR THE LATER PARTS OF THE SAME PROBLEM.

GOOD LUCK!

An overview of the exam problems. Each problem is worth 12 points.
Each part of a problem is weighted equally.
Take a few minutes to look at all the questions, BEFORE you start solving.

1. a) Write statements (i), (ii), and (iii) below in formal logic, using the notation

p : "I practice a lot" r : "I win the race" t : "My teammate wins the race"
o : "I go to the olympics" m : "We get a medal".

(i) If I practice a lot, then I will win the race and go to the olympics
(ii) If my teammate or I wins the race, then we get a medal.
(iii) I practice a lot or my teammate wins the race.

b) Given statements (i–iii) above, prove (in formal logic) that we get a medal.
c) Given statements (i–iii) above, prove (again in formal logic) that:
If I do not go to the olympics, then my teammate wins the race.

2. Let $f : X \to Y$ be a function, and let $A \subseteq X$.
a) Suppose f is injective. Prove that $f(\overline{A}) \subseteq \overline{f(A)}$. (Reminder: $f(\overline{A}) = f(X - A)$, but $\overline{f(A)} = Y - f(A)$.)
b) Suppose instead that f is surjective. Prove that $\overline{f(A)} \subseteq f(\overline{A})$.

3. We define the function $f : \mathbf{Z} \to \mathbf{Z}$ by $f(n) = \lfloor \frac{n}{4} \rfloor$.
a) Find $f(n)$ for every $n \in \{0, 1, 2, 3, 4, 8, 9, 10, 11, 12\}$.
b) Show that $\forall b \in \mathbf{Z}$, $\left|f^{-1}(\{b\})\right| = 4$. (Hint: list the elements of $f^{-1}(\{b\})$.)
c) Is f injective? surjective? Explain your answers.
d) Find the cardinality $|X|$ of the set

$$X = \{A \subseteq \mathbf{Z} \mid f(A) \subseteq \{0, 1, 2, 3, 4, 5, 6\} \wedge |f(A)| = 3\}.$$

4. The Tutti Frutti Bakery makes manaqish in five flavors: zaatar, apple, banana, cherry, and date. How many ways are there to buy 20 manaqish such that you have as many zaatar manaqish as you like, but no more than 5 each of apple, banana, cherry, and date? (For example, it is possible to buy 4 zaatar, 5 apple, 5 banana, 4 cherry, and 2 date; it is also possible to buy 19 zaatar, 0 apple, 1 banana, 0 cherry, and 0 date.)

5. a) Prove that $\lceil x \rceil^2$ is $\Theta(x^2)$.
b) Prove that $2^{(\lceil x \rceil^2)}$ is $O(4^{(x^2)})$.
c) Prove that $2^{(\lceil x \rceil^2)}$ is NOT $O(2^{(x^2)})$. (Hint: what happens when $x = n + 1/2$ for $n \in \mathbf{N}$?)

6. a) Find the prime factorization of each of 2900 and 1240, and use this to deduce $gcd(2900, 1240)$.
b) (UNRELATED) Use the extended Euclidean algorithm to find $x_0, y_0 \in \mathbf{Z}$ such that $67x_0 + 23y_0 = 1$.
c) Let $m \in \mathbf{Z}$. Use the result of part (b) to show that $(23 \mid m) \wedge (67 \mid m)$ **if and only if** $(23 \cdot 67) \mid m$. Hint: $m = m \cdot 1$. Note that you can answer this problem without solving for the specific values of x_0, y_0.

7. a) Let $i, j \in \mathbf{N}$ satisfy $i \geq 1 \wedge j \geq 1$. **Fix** i, and **prove by induction on** j that

$$ij \geq i + j - 1.$$

(Remark: there is an easier way to prove the above by expanding $(i-1)(j-1) \geq 0$. **However, I want you to prove the result using mathematical induction.**)

b) Define a sequence $a_1, a_2, a_3, \ldots$ recursively by:

$$a_1 = 1; \qquad \forall n \geq 2, \quad a_n = \frac{\sum_{k=1}^{n-1} a_k a_{n-k}}{n} = \frac{a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1}{n}.$$

Find the values of a_2, a_3, and a_4. (DO simplify the fractions.)

c) Use strong induction to prove: $\forall n \geq 1, \quad 0 \leq a_n \leq \frac{1}{n}$. Note that part (a) is useful here.

8. Answer the following exercise in GAP or in pseudocode. You are **not** allowed to use high-level builtin GAP functions such as `ForAny()` or `Filtered()`. However, you **are** allowed to use `IsEmpty()`, `addfirst()`, `addlast()`, `removefirst()`, and `removelast()` just as in the example file `example3.txt` on recursion. You may also use the functions `IsOddInt()`, `QuoInt()`, and `RemInt()`. Reminders:

```
addfirst([3,1,4,1,5,9], 211) = [211,3,1,4,1,5,9]
addlast([3,1,4,1,5,9], 211) = [3,1,4,1,5,9,211]
removefirst([3,1,4,1,5,9]) = [1,4,1,5,9]
removelast([3,1,4,1,5,9]) = [3,1,4,1,5]
IsOddInt(17) = true,      IsOddInt(18) = false
QuoInt(17,3) = 5,         RemInt(17,3) = 2       # because 17 = 5*3 + 2
```

a) Write a **NONrecursive** function `countodd(ll)` that returns the number of odd elements of the list `ll`. For example, `countodd([3,1,4,1,5,9]) = 5` because of the five odd elements `3, 1, 1, 5, 9`. This problem should use a loop, and no recursion.

b) Write a **RECURSIVE** version `countoddR(ll)` of the same function as above. No loops allowed! Reminder: `ll[1]` is the first element of `ll`.

c) (UNRELATED) Write a **RECURSIVE** function `baseb(N,b)` that returns a list `[ak,...,a1,a0]` of digits of `N` in base `b`. Here each `ai` is an integer with `0 <= ai < N`, and we have `N` $= (a_k \cdots a_1 a_0)_b$. For example: $13 = (1101)_2$ and $42 = (2A)_{16}$, so we want `baseb(13,2) = [1,1,0,1]` and `baseb(42,16) = [2,10]`. You may assume that `N >= 0` and that `b >= 2`. Once again, no loops allowed!

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